MONOTONE EQUILIBRIA IN GAMES WITH MAXMIN EXPECTED UTILITY

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Abstract. I introduce ambiguity aversion to a class of games that includes the all-pay auction and war of attrition. The main result is a characterization of the set of increasing equilibria. Unlike with subjective expected utility, even when beliefs are independent of type, an increasing equilibrium may not exist. Sufficient conditions are provided for such an equilibrium to exist. The games are compared in terms of the expected sum of expenditures.

1. Introduction

Since the work of Ellsberg[11], there has been a large body of experimental work that demonstrates that some behavior under uncertainty cannot be explained by maximization of subjective expected utility (SEU). Behavior that contradicts SEU is particularly common when there is ambiguity about the probability of events. In the finance and macroeconomics literature, ambiguity aversion, which generalizes SEU, has been shown to solve many puzzles regarding asset prices, notably the equity premium puzzle.[1] Mukerji[29] has shown that ambiguity aversion can explain incompleteness of contracts in situations where costly contracting alone gives counter-factual predictions. Kagel and Levin[18] proposed ambiguity aversion as an explanation for overbidding in first-price auctions.

I apply ambiguity aversion to a class of games of incomplete information where the all-pay auction[32] and war of attrition[26] are limiting cases[43][31]. These games have been used to model a wide variety of strategic environments in which players compete for a prize by expending resources or effort. Examples include firms competing to determine industry standards[8], firms exiting a crowded market[13], students competing for college admissions[17], and online auctions[35].

Ambiguity is likely to be present in many environments modeled by games of incomplete information. In the games studied here, players are uncertain about how much another

Date: December 26, 2013.
I would like to thank Aaron Bodoh-Creed, Alan Gelder, Dan Kovenock, and Nicholas Yannelis for helpful discussion and suggestions. I would also like to thank participants of the SAET Conference 2013, Midwest Economic Theory Conference Spring 2013, Iowa Alumni Workshop 2012, and University of Iowa Seminars.

[1]See Guidolin and Rinaldi[15] for a recent survey. See also Hansen and Sargent[16].

player values a prize. With SEU, this uncertainty is modeled using a distribution over the other player's value called a belief distribution. The player's interim utility is the expected utility calculated using that distribution. However, in many applications the distribution of the other player's value is hard to learn either because the environment is changing or because of a lack of experience. Following Lo[25], I model ambiguity using the maxmin expected utility model of Gilboa and Schmeidler[14]. With MEU, beliefs are modeled by a set of distributions, Δ, of which any one may generate the other player's value. The player calculates the expected utility for each distribution in Δ, and the utility is the lowest expected utility for any distribution in Δ. By choosing an action which maximizes the minimum expected utility an ambiguity averse player chooses an action which is robust to the worst case distribution.

I find that ambiguity has a significant impact on both the expected sum of expenditures and the efficiency of the games studied here. With MEU, players expend more resources in the first-price auction than in the all-pay auction. Under conditions that insure the existence of an increasing equilibrium, players expend more in the all-pay auction than in the war of attrition. Also, these games may fail to have an efficient equilibrium in the sense that the prize is not always awarded to the player who values it the most. As I explain in the literature review, these results contrast the results in an analogous SEU environment.

I provide a characterization for the increasing, symmetric equilibria of games in this class. Previously, a general characterization of equilibrium was only available for games like the first-price auction where the minimizing distribution is the one which minimizes the probability of winning. Because the games studied here may not have that property, a different method is required to derive the equilibrium. The technique for characterizing equilibria can be applied to other games with MEU. I also provide conditions for an increasing equilibrium to exist.

1.1. **Related Literature.** In his seminal work, Lo[25] applied the MEU model to the first-price auction where he derived the unique increasing equilibrium. His analysis depends on the following observation about the first-price auction. Any bid submitted by a player results in one of two outcomes; either the player wins the object and pays the bid, or the player does not win the object and makes no payment. Because of this, assuming that players bid below their own value, an expected utility minimizing distribution is a distribution in Δ which minimizes the probability of having the highest bid. Thus, the set of expected utility minimizing distributions does not depend on the player's value. As Lo[25] noted, in many auctions the expected utility depends on more than just the probability of having the highest bid. More generally, the minimizing distribution may also depend on the player's own value if the ex post utility depends on more than whether the player wins and the player's own bid.

Since the work of Lo[25], there have been a number of papers that apply MEU to a variety of games and mechanism design environments. These games and mechanisms have the property that the ex post utility, given a bid, only depends on winning or losing. Levin and Ozdenoren[24] study the first-price auction with particular emphasis on uncertainty about the number of bidders. Again, the minimizing distribution is the one which minimizes the probability of winning. Bose, Ozdenoren, and Pape[7] and Bodoh-Creed[6] consider the problem of designing revenue maximizing mechanisms. Both of the mechanism design papers find that, in the optimal mechanism, the set of minimizing distributions is the same as in the first-price auction and does not depend on the type of the player.

In the war of attrition, the worst case distribution may depend on the player's type. In this game the player with the highest bid wins the object and pays the losing bid. The loser pays his own bid and does not receive a prize. Thus the ex post utility in the war of attrition depends on whether the player wins or not, and for the winner, it depends on the losing bid. This makes the war of attrition much different from the first-price auction when there is MEU. The expected utility will depend on both the probability of winning and the expected payment. The relative importance of these two parts will depend on how much the player values the prize. Players with high values will tend to prefer distributions with a higher probability of winning, whereas players with low values will be more concerned with the expected payment. Thus the distribution in Δ that minimizes a player's expected utility may be different for types with different values. As a result, a different method of analysis is needed from the one used for first-price auctions. One consequence is that, whereas the first-price auction has an increasing equilibrium in the symmetric MEU model, this may not be the case with the war of attrition.

The existence of an increasing equilibrium in the war of attrition and all-pay auction has been studied with SEU. An equilibrium exists when the players' values are distributed independently so that each player's belief does not depend on the player's own type[2][3][43]. In the affiliated private values model with players who have SEU, Milgrom and Weber[28] show that to have an increasing equilibrium, in the first-price auction, it is sufficient that a player's belief is increasing in the player's value, in the sense of affiliation. Krishna and Morgan[23] show that, when players have SEU, an increasing equilibrium exists in the war of attrition and all-pay auction if the affiliation is moderate.

In the MEU model, even when the player's belief, Δ, is independent of the player's value, an increasing equilibrium may not exist. This is because the expected utility minimizing distribution may depend on the player's value. The dependence arises because in games like the war of attrition the payment, in the event of winning, depends on the other player's bid. Since the expected payment depends on the other player's strategy, the set of minimizing distributions depends on the equilibrium. This contrasts Krishna and Morgan[23] where the distribution used to calculate expected utility depends on the value exogenously through Bayesian updating.

The revenue equivalence theorem[30] implies that when players have SEU and independent beliefs the sum of expenditures in the games studied here is the same as in the first-price and second-price auctions. With a particular form of Δ, Lo[25] showed that the first-price auction generates more revenue than the second-price auction with MEU. Bodoh-Creed showed that more generally no revenue ranking exists between the two auctions when players have MEU. The ranking that I provide for the all-pay auction and first-price auction holds quite generally; the ranking of the all-pay auction and war of attrition holds under a set of conditions that insure that an increasing equilibrium exists. Thus, these revenue rankings are the most general rankings that I know of for commonly used mechanisms. It should be noted that these rankings are the reverse of what Krishna and Morgan[23] find in the affiliated values SEU environment. The ranking between the first-price and all-pay auction complements Fibich, Gavious, and Sela[12] who study the independent values environment with risk averse players who have SEU preferences.

The remainder of the paper is organized as follows. Section 2 formally presents the model. Section 3 discusses games in which the set of minimizing distributions is independent of the value. These games can be ranked by the expected sum of expenditures. Section 4 contains a characterization of the set of symmetric, increasing equilibria in the general model. Sufficient conditions for existence of such an equilibrium are provided. Section 5 provides an example that illustrates why an ex post efficient equilibrium may not exist with MEU. Section 6 discusses some extensions including the smooth ambiguity aversion model and type dependent ambiguity. Section 7 concludes. The appendix contains some of the proofs.

2. Model

I begin by describing the class of games studied here. Although all of the results apply to games with any finite number of players, to simplify notation, I consider games with two players, player 1 and player 2.[2] Player i's value for a prize, v_i, comes from the interval $[\underline{v}, \overline{v}]$, with $\underline{v} \geq 0$. Upon learning their own values, both players simultaneously submit a bid. Let

[2]The main savings occurs when I describe beliefs. With multiple players the relevant distribution for calculating expected utility is the distribution of the highest bid of the $n-1$ other players.

$b_i \geq 0$ be player i's bid. The allocation function, $x_i(b_1, b_2)$, determines the probability that player i receives the prize. In addition, each player i has a transfer function, $\tau_i(b_1, b_2)$, which is player i's expenditure as a function of the bids.

I will restrict attention to a specific, well studied class of allocation and transfer functions which includes the all-pay auction and war of attrition[2].

$$x_i(b_1, b_2) = \begin{cases} 1, \text{ if } b_i > b_j \\ 1/2, \text{ if } b_i = b_j \\ 0, \text{ otherwise} \end{cases} \tag{1}$$

$$\tau_i(b_1, b_2) = \begin{cases} (1-p)b_i + pb_j, \text{ if } b_i > b_j \\ b_i, \text{ otherwise} \end{cases} \tag{2}$$

Where $p \in [0, 1)$. When $p = 0$, the transfer function is that of the all-pay auction. When $p = 1$, the game is a static version of the war of attrition. The usual interpretation of the war of attrition is that players expend resources over time until one side concedes at which point the game terminates instantly. In a dynamic setting, $p < 1$ captures a situation in which a player does not learn about an opponent's concession immediately[2]. p could also be thought of as the probability that the winner pays the losing bid.[3]

The ex post utility of player i is given by the following utility function.

$$u_i(b_1, b_2, v_i) = x_i(b_1, b_2)v_i - \tau_i(b_1, b_2). \tag{3}$$

This utility is known as the risk-neutral, private values model. To better understand the role of ambiguity aversion I restrict attention to this benchmark model of ex post utility.[4]

Now I define the interim utility, which is the utility at the point where each player knows his own value but not the other player's value. This is modeled by the maxmin expected utility of Gilboa and Schmeidler[14]. A player's ambiguous belief is a set of distributions over the other player's value. The player's utility is the lowest expected utility generated by any distribution in the set.

Let $\Delta_i(v_i)$ be the belief set for player i with a value of v_i. The expected utility, with respect to distribution G, for player i with value v_i and bid b_i is defined as

$$\tilde{U}_i(b_i; v_i, s, G) \equiv \int_{\underline{v}}^{\overline{v}} u_i(b_i, s(v_j), v_i) g(v_j) dv_j \tag{4}$$

[3]See Bulow and Klemper[8] for some natural ways to extend this description to more than two players.

[4]This model can handle asymmetric information regarding both costs and values. That is, the model also includes the seemingly more general case that $\tilde{u}_i(b_1, b_2, v_i, c_i) = x_i(b_1, b_2)v_i - c_i\tau_i(b_1, b_2)$ where c_i is interpreted as the marginal cost of expenditure. Using an affine transformation $\tilde{u}$ becomes $u_i(b_1, b_2, v_i/c_i) = x_i(b_1, b_2)v_i/c_i - \tau_i(b_1, b_2)$, which is equivalent to the original model as long as $c_i > 0$.

where $s : [\underline{v}, \overline{v}] \to R^+$ is a measurable strategy and G is a distribution over $[\underline{v}, \overline{v}]$, with density g. The maxmin expected utility is given by

$$\tilde{V}_i(b_i; v_i, s) \equiv \inf_{G \in \Delta_i(v_i)} \tilde{U}(b_i; v_i, s, G). \tag{5}$$

If $\Delta_i(v_i)$ is singleton, then the utility corresponds to SEU; otherwise, the player is said to be ambiguity averse.

An equilibrium of the game is analogous to an equilibrium in a game with subjective expected utility; each player's strategy maximizes his utility given the other player's strategy.

Definition 2.1. A pure strategy equilibrium is a pair of measurable strategies (s_1, s_2) such that for each player $i \in \{1, 2\}$ and $j \neq i$ and for every $v_i \in [\underline{v}, \overline{v}]$,

$$\tilde{V}_i(s_i(v_i), v_i; s_j) \geq \tilde{V}_i(b_i, v_i; s_j)$$

for all $b_i \geq 0$.

I will further restrict attention to equilibria which are increasing. For the rest of the paper, when I refer to an increasing strategy I mean a strategy which is strictly increasing in the player's value. That is if $v > v'$, then $s(v) > s(v')$ for all $v, v' \in [\underline{v}, \overline{v}]$. There are several reasons for focusing on increasing strategies. First, when the environment is symmetric, the symmetric, increasing equilibria are the ex post efficient equilibria. Also, if Δ is singleton the unique equilibrium is in increasing strategies[2][34].

For an increasing strategy s and for each bid $b \in [s(\underline{v}), s(\overline{v})]$, there is at most one value $z \in [\underline{v}, \overline{v}]$ such that $b = s(z)$. Thus, it will be notationally convenient to think of the players as choosing the value that corresponds to a bid rather than choosing a bid. If a player chooses bid b this is equivalent to choosing $z = s^{-1}(b)$ when the other player is using the continuous, increasing strategy s. Now, with continuous, increasing strategies and this notation, write the expected utility of a player with value v_i who bids $s_j(z)$ as

$$U_i(z; v_i, s, G) \equiv \tilde{U}_i(s_j(z); v_i, s_j, G) = v_i G(z) - p \int_{\underline{v}}^{z} s_j(t) g(t) dt - (1 - pG(z)) s_j(z). \tag{6}$$

This follows since with an increasing strategy the probability that a player pays his own bid is given by $1 - pG(z)$. With the complementary probability the player pays the other player's bid. From the point of view of a player, the other player's bid is a random variable determined by the strategy and distribution of the other player's value. The maxmin expected utility is similarly defined as a function of z and is denoted by $V_i(z; v_i, s)$.

A. 1. $\Delta_i(v_i) = \Delta$ for all $v_i \in [\underline{v}, \overline{v}]$, for $i = 1, 2$.

This assumption makes the model as close as possible to the benchmark SEU model in which values are independent and identically distributed. A.1 says that the beliefs for both players are the same and that those beliefs do not depend on the player's value. By focusing on this simple model the role of ambiguity aversion is most transparent. I discuss making the beliefs of the players dependent on the player's value in Section 6. Because I focus on a symmetric environment the player subscript is often omitted.[5]

A. 2. Δ is a set of continuously differentiable, strictly increasing distributions on $[\underline{v}, \overline{v}]$. Δ is convex and compact in the sense that the densities form a compact set with respect to the uniform topology.[6]

At this stage, note that A.2 insures that there is a distribution in Δ that minimizes the expected utility. This follows because the expected utility will be continuous in the distribution of the other player's value. Throughout this paper, A.1 and A.2 are implicitly assumed unless otherwise stated.

3. Type Independent Minimizing Distributions

3.1. **Equilibrium.** To begin, I derive the equilibrium in the games for which the set of minimizing distributions does not depend on the player's value. To discuss such games, the following notation is useful. Define the absolutely continuous distribution $F_m(v) \equiv \min_{G\in\Delta} G(v)$ for all $v \in [\underline{v}, \overline{v}]$. F_m is the lower envelope of the distributions in Δ; in general, F_m may not be contained in Δ.

First, consider the all-pay auction, which is similar to the first-price auction in the sense that the ex post utility given a bid only depends solely on having the highest bid. As in the first-price auction, in the all-pay auction, the set of minimizing distributions does not depend on the player's value. This is true because in the all-pay auction, an expected utility minimizing distribution minimizes the probability of winning. For an increasing strategy s, the MEU is written

$$V(z; v, s) = \min_{G\in\Delta} vG(z) - s(z) = vF_m(z) - s(z). \tag{7}$$

The second equality follows since $F_m(v)$ coincides with the lowest probability of having the highest bid. This observation leads to the following equilibrium characterization. [7]

[5]For a discussion of asymmetry with SEU see Amann and Leininger[2]

[6]For continuously differentiable distributions this is the same as the topology induced by the norm $\max\left\{\sup_{v\in[\underline{v},\overline{v}]} |F(v)|, \sup_{v\in[\underline{v},\overline{v}]} |f(v)|\right\}$(Abbott p.164 2001[1], Rudin p.152 1976[37]).

[7]That the first-price and all-pay auctions have the same set of minimizing distributions was briefly noted in Bodoh-Creed[6].

Lemma 3.1. *The unique increasing equilibrium of the all-pay auction is for players to use the strategy*

$$\beta(v) = \int_{\underline{v}}^{v} t f_m(t) dt. \tag{8}$$

Proof: The proof follows from the observation that the players' utility is the same as in the game where F_m is the unique distribution in the players' belief set (i.e. SEU). The uniqueness and characterization follow from the standard analysis of the all-pay auction in Amann and Leininger[2]. □

Unlike in the all-pay auction, when $p \neq 0$, the expected utility minimizing distribution may not minimize the probability of winning. This is because the payment in the event of winning depends on the other player's bid. However, if $F_m \in \Delta$, F_m is always one of the minimizing distributions. This follows since F_m first order stochastic dominates (FOSD) the distributions in Δ, and the payment for a player is weakly increasing in the other player's bid. Thus F_m both maximizes the expected payment and minimizes the probability of winning.[8]

A. 3. $F_m \in \Delta$.

Using assumption A.3 the utility can be written

$$V(z; v, s) = vF_m(z) - p \int_{\underline{v}}^{z} s(t) f_m(t) dt - (1 - pF_m(z))s(z). \tag{9}$$

This leads to the following equilibrium characterization.

Lemma 3.2. *A.3 implies that the unique symmetric, increasing equilibrium, when $p \in [0, 1)$,*[9] *is for players to use the strategy*

$$\beta(v) = \int_{\underline{v}}^{v} \frac{t f_m(t)}{1 - pF_m(t)} dt. \tag{10}$$

Proof: The proof is the same as Lemma 3.1.□

[8] The FOSD assumption has been used by Stong[40] to develop comparative statics with regard to the size of Δ. Lo[25] and Bodoh-Creed[6] used sets satisfying this property to study the revenue ranking of the first- and second-price auctions. Levin and Ozdenoren[24] use the stronger monotone likelihood ratio order to prove results when there is an uncertain number of bidders in the first-price auction.This assumption is also used to prove results in the mechanism design literature[7]. Carvalho[9] uses the stronger reverse-hazard-rate order to establish results in auctions with smooth ambiguity aversion.

[9] Nalebuff and Stiglitz[32] show that in the war of attrition there is a continuum of asymmetric equilibria. The equilibrium described is the unique symmetric equilibrium when $p = 1$.

3.2. **Revenue.** In this section, I provide a revenue ranking for the class of games studied here as well as a comparison to the first-price auction. The term revenue is in keeping with the auction literature; however, in many applications, the expenditure of resources may be wasteful to society. The classical example is rent seeking behavior as studied by Tullock[42].

To calculate expected revenue, suppose that both players' values are drawn independently from a distribution $F : [\underline{v}, \overline{v}] \to [0,1]$. F could be thought of as the belief of an ambiguity neutral seller.[10] In order to be able to discuss general revenue rankings, it is necessary to impose a consistency requirement on the beliefs of the players and the seller. Otherwise, F could be chosen to produce any revenue ranking. A.4 below is a generalization of the common prior assumption. The assumption says that the players and the seller agree in the sense that the seller's belief is in the players' ambiguous belief set.

A. 4. $F \in \Delta$.

Proposition 3.3. *Given assumption A.4, the expected revenue of the first-price auction is higher than that of the all-pay auction.*

Proof: Lo[25] shows that

$$\hat{B}(v) = v - \frac{\int_{\underline{v}}^{v} F_m(t)dt}{F_m(v)} \tag{11}$$

is the unique equilibrium of the first-price auction. Let $e_f(v)$ be the expected expenditure of a player with value v in the first-price auction.

$$e_f(v) = \left(v - \frac{\int_{\underline{v}}^{v} F_m(t)dt}{F_m(v)} \right) F(v) \tag{12}$$

Let $e_a(v)$ be the expected expenditure of a player with value v in the all-pay auction. Since the player always pays his bid, integration by parts implies

$$e_a(v) = \beta(v) = vF_m(v) - \int_{\underline{v}}^{v} F_m(t)dt. \tag{13}$$

Applying these formulas

$$e_f(v) - e_a(v) = v\left(F(v) - F_m(v)\right) + \left(\frac{F_m(v) - F(v)}{F_m(v)} \right) \int_{\underline{v}}^{v} F_m(t)dt \tag{14}$$

[10] All of the results presented in this section can be extended to the case of a seller with MEU. If Δ_s is the seller's belief, it is sufficient that there is a distribution $F^M \in \Delta_s$ which is dominated in the sense of FOSD by all other distributions in Δ_s, and $\Delta_s \cap \Delta \neq \varnothing$. Since I focus on increasing strategies F^M minimizes the sellers revenue and the proofs go through substituting F^M for F. See Lo[25] and Bose, Ozdenoren, and Pape[7] for more discussion of ambiguity averse sellers.

$$= (F(v) - F_m(v)) \left(v - \frac{\int_{\underline{v}}^{v} F_m(t)dt}{F_m(v)} \right) \tag{15}$$

$$= (F(v) - F_m(v))\hat{B}(v) \geq 0. \tag{16}$$

The inequality follows from the fact that F_m FOSD F. □

The intuition for this result is the following. A player, with value v, in the all-pay auction, makes a payment which is deterministic since the payment does not depend on the other player's bid. On the other hand, in a first-price auction the player's payment is conditional on having the highest bid. The revenue equivalence theorem[30][36] implies that if the seller's belief and the players' minimizing distribution coincide (i.e. $F_m = F$), the revenue is the same for both games. If $F(v) > F_m(v)$, the probability of the player paying the bid in the first-price auction is higher for F. Thus, the expected payment is higher in the first-price auction from the seller's perspective.

Proposition 3.4. *Given assumptions A.3 and A.4, the expected revenue is decreasing in p.*

Proof: Let $e_p(v)$ be the expected expenditure of a player with value v in the game with parameter p.

$$e_p(v) = p \int_{\underline{v}}^{v} \beta(t) f(t)dt + (1 - pF(v))\beta(v)$$

$$= \beta(v) - p \int_{\underline{v}}^{v} \beta'(t)F(t)dt$$

$$= \int_{\underline{v}}^{v} \frac{tf_m(t)}{1 - pF_m(t)}dt - p \int_{\underline{v}}^{v} \frac{tf_m(t)}{1 - pF_m(t)}F(t)dt$$

$$= \int_{\underline{v}}^{v} \frac{tf_m(t)}{1 - pF_m(t)}(1 - pF(t))dt$$

This is nonincreasing in p since F_m FOSD F. The second line is by integration by parts and the third is by applying the formula for the equilibrium.□

The intuition for this result is similar to the argument for Proposition 3.3. When $F(v) = F_m(v)$ the revenue equivalence theorem implies that the revenue is the same for all values of p. However, when $F(v) > F_m(v)$, a player with value v is more likely to have the highest bid. When the player wins it is more likely to pay the highest bid if p is low. It follows that the expected payment is decreasing in p.

In summary the first-price auction produces more revenue than the all-pay auction, and when A.3 is satisfied, the all-pay auction produces more revenue than the symmetric equilibrium of the war of attrition. The format that is preferred will depend on whether the bid is interpreted as a productive or wasteful use of resources. The payoff equivalence theorem given in Bodoh-Creed[6] implies that players are indifferent between first-price and all-pay auctions, and with assumption A.3, the players are indifferent between all of the games. Thus in applications where the players' expenditure is wasteful to society the war of attrition may Pareto dominate the other forms of competition discussed.

4. Type Dependent Minimizing Distributions

In the previous section, the equilibrium characterization for games with $p \neq 0$ was facilitated by the assumption that there is a distribution in Δ which first order stochastic dominates the others. With this assumption the worst case distribution does not depend on the player's value. However, for some applications, A.3 may be too restrictive. If Δ is interpreted as a belief, there are many natural ways for the belief to be formed that would not satisfy this assumption.

For instance, A.3 may not be appropriate when players are uncertain about the dispersion in the distribution of types.[11] One application in which the level of variance may be ambiguous is labor strikes. Kennan and Wilson[19] model labor strikes as war of attrition in which each side is uncertain of the other side's cost of conceding in a labor dispute. The variance of the firms cost can be affected by market conditions or by decisions made by the managers.[12] If the union faces ambiguity about the variance of the firms costs the first order stochastic dominance assumption may not adequately model the union's information.

4.1. **Characterization.**

I show that with a general form of Δ the set of equilibria can be analyzed in a way analogous to games with SEU. With SEU, an increasing equilibrium must be a solution to a differential equation. This equation is defined by observing that the derivative of the expected utility must be zero at the equilibrium bid. There are a couple of difficulties to overcome to apply this method to MEU. The first is to show that the MEU is sufficiently differentiable. This requires an envelope theorem which insures differentiability and gives a formula for the derivative. I show that in equilibrium MEU is right-hand and left-hand differentiable almost everywhere, and that in some sense, which will be made precise,

[11]Konrad and Kovenock[21] study the effects of variability in the distribution of types on behavior for some contest environments.

[12]There is a growing literature that studies the strategic importance of risk taking. See Kräkel[22] and Suzuki[41] and the references therein for recent references.

the derivative can be set to zero at the equilibrium bid.[13] Since the derivative of the utility is not uniquely defined at some points, it is useful to think of an equilibrium strategy as a solution to a differential inclusion.

For this section, assumption A.2 is used to insure that the set of minimizing distributions has convenient properties. Since the expected utility is continuous in the distribution and Δ is compact, the set of minimizers $\mathcal{G}_{v,z,s} \equiv \{G \in \Delta : U(z; v, s, G) = V(z; v, s)\}$ is nonempty. The convexity of Δ insures that $\mathcal{G}_{v,z,s}$ is convex valued. Additionally, it is convenient to note that the compactness of Δ implies that all of the densities of the distributions in Δ have a common upper bound $\overline{g}$.

As a matter of notation, the subscript on the player's value is often suppressed since I will be focused on a symmetric environment. A selection of the correspondence $\mathcal{G}_{v,z,s}$ is a function $G_s : [\underline{v}, \overline{v}] \times [\underline{v}, \overline{v}] \to \Delta$ such that $G_{v,z,s} \in \mathcal{G}_{v,z,s}$ for all $v, z \in [\underline{v}, \overline{v}]$.

The first step to characterizing an equilibrium is to establish the differentiability of the equilibrium utility. One can only hope to establish differentiability of the utility if players are using sufficiently smooth strategies. The following lemma says that any increasing, symmetric equilibrium strategy is a Lipschitz continuous function.

Lemma 4.1. *If β is an increasing, symmetric equilibrium strategy, then β is Lipschitz continuous with constant $M = \overline{v}\overline{g}/(1-p)^2$.*

In the proof, it is shown that if the strategy is not Lipschitz continuous, there is a profitable deviation for a positive measure of values. Since the equilibrium strategies are Lipschitz continuous, I restrict attention to such strategies without any loss of generality.

The following envelope condition establishes the differentiability of the minimum expected utility, V, and provides a formula for the derivative.

Proposition 4.2. *[Envelope Theorem] Let $s(v)$ be a strictly increasing strategy which is Lipschitz continuous with constant $M = \overline{v}\overline{g}/(1-p)^2$.*

(1) *$V(z; v, s))$ is absolutely continuous in z.*
(2) *V is right-hand and left-hand differentiable in z at v for almost all $v \in [\underline{v}, \overline{v}]$.*
(3) *Let $G_{v,z,s} \in \mathcal{G}_{v,z,s}$ be given. If $z > \underline{v}$ and $V(.; v, s)$ is left-hand differentiable at z, then $V'_{-}(z; v, s) \geq U'(z; v, s, G_{v,z,s})$. If $z < \overline{v}$ and $V(.; v, s)$ is right-hand differentiable at z, then $V'_{+}(z; v, s) \leq U'(z; v, s, G_{v,z,s})$. If $z \in (\underline{v}, \overline{v})$ and $V(.; v, s)$ is differentiable at z, then $V'(z; v, s) = U'(z; v, s, G_{v,z,s})$.*

[13]All references to a measure refer to the Lebesgue measure.

The proof applies the envelope theorems in Milgrom and Segal[27]. If $V(.;v,s)$ is differentiable at z, the derivative of the expected utility is given by

$$V'(z;v,s) = vg_{v,z,s}(z) - (1 - pG_{v,z,s}(z))s'(z). \tag{17}$$

Setting this derivative to zero at the equilibrium bid motivates the expression in the statement of Theorem 4.3. This is only a heuristic motivation because the envelope theorem *does not* say that the MEU is differentiable at the equilibrium bid.

Theorem 4.3. *If β is an increasing, symmetric equilibrium, then there exists a selection $G_{z,v,\beta} \in \mathcal{G}_{z,v,\beta}$ for all $z, v \in [\underline{v}, \overline{v}]$ such that*

$$\beta(v) = \int_{\underline{v}}^{v} \frac{tg_{t,t,\beta}(t)}{1 - pG_{t,t,\beta}(t)}. \tag{18}$$

The main idea of the proof is that, for a strategy to be a symmetric equilibrium, the equilibrium bid must be a local maximum. The envelope theorem implies that the left-hand derivative of V is non negative and the right-hand derivative is non positive in equilibrium for almost all values of v. The convexity of the set of minimizing distributions implies that, at almost all equilibrium bids, there is a minimizing distribution, $\check{G}$, such that the derivative of the expected utility, $U'(v;v,\beta,\check{G})$, is zero. It is in this sense that, at the equilibrium bid, the derivative is equal to zero almost everywhere.

The condition in the theorem is only a necessary condition. *A priori*, there seem to be two possible ways for an efficient equilibrium to fail to exist. One is that all strategies satisfying the necessary condition do not identify a global maximum. The other is that there may fail to be a strategy that satisfies (18). Since the set of minimizing distributions depends on the strategy played, and the strategy is defined using a selection from the set of minimizing distributions it is not obvious that such a strategy exists.

The existence of a solution to the necessary condition can be understood as the existence of a solution to the following differential inclusion.

$$\beta'(v) \in \left\{ \lambda \in \mathbf{R} : \lambda = \frac{vg_{v,v,\beta}(v)}{1 - pG_{v,v,\beta}(v)} \text{ for } G_{v,v,\beta} \in \mathcal{G}_{v,v,\beta} \right\} \tag{18$'$}$$

A restatement of Theorem 4.3 is that an equilibrium strategy must be a solution to this differential inclusion.[14] As stated in Proposition 4.4, there is always a solution to (18′). That is, there is always a strategy which satisfies the necessary condition. Thus, if there is

[14]The usual form of a differential inclusion problem is to find an absolutely continuous function $x : I \to \mathbf{R}$ such that $x'(t) \in F(x(t),t)$ for almost all t in an interval I where $F : \mathbf{R} \times I \to 2^{\mathbf{R}}$ is potentially multivalued. In this case, I seek a solution to an inclusion of the form $x'(t) \in \hat{F}(x(t),t,x(.))$ where $\hat{F} : \mathbf{R} \times I \times C(I) \to 2^{\mathbf{R}}$.

no efficient equilibrium it is because every solution to the necessary condition fails to identify a global maximum.

Proposition 4.4. *There exists a strategy* β *and a selection* $G_{z,v,\beta} \in \mathcal{G}_{z,v,\beta}$ *for all* $z, v \in [\underline{v}, \overline{v}]$ *such that*

$$\beta(v) = \int_{\underline{v}}^{v} \frac{t g_{t,t,\beta}(t)}{1 - pG_{t,t,\beta}(t)}.$$

4.2. **Existence.** Existence of an equilibrium can be established by checking each strategy in the set of strategies that satisfy the necessary condition. For each strategy that satisfies the characterization in Theorem 4.3, the utility of submitting a bid can be rewritten under the assumption that the other player uses the candidate strategy. Let β and $G_{v,z,\beta}$ be as in Theorem 4.3.

$$V(z; v, \beta) = vG_{v,z,\beta}(z) - p\int_{\underline{v}}^{z} \beta(t) g_{v,z,\beta}(t)dt - (1 - pG_{v,z,\beta}(z))\beta(z)$$

$$= vG_{v,z,\beta}(z) + p\int_{\underline{v}}^{z} \beta'(t) G_{v,z,\beta}(t)dt - \beta(z)$$

$$= \int_{\underline{v}}^{z} \left\{ \frac{v g_{v,z,\beta}(t)}{1 - pG_{v,z,\beta}(t)} - \frac{t g_{t,t,\beta}(t)}{1 - pG_{t,t,\beta}(t)} \right\} (1 - pG_{v,z,\beta}(t))dt \tag{19}$$

The second line follows from integration by parts and the third follows by applying (18).

This is similar to an expression that arises in Krishna and Morgan[23] with subjective expected utility and affiliated distributions. By making assumptions about the unique prior distribution of values they insure that (19) is increasing in z for $z < v$ and decreasing for $z > v$. Thus it seems natural to look for conditions under which the same is true with MEU. However, is it is difficult to establish quasi-concavity of (19) because the minimizing distribution may depend on v and this dependence is endogenous.

Proposition 4.5 below gives a condition under which, in equilibrium, the minimizing distribution is independent of the player's value. A.3 implies that the minimizing distribution is independent of the player's value regardless of the increasing strategy that the other player uses. Proposition 4.5 gives a weaker condition than FOSD that insures that, in equilibrium, the minimizing distribution is the same for any bid and value.

Proposition 4.5. *If there is a* $G^* \in \Delta$ *such that for all* $G \in \Delta$ *and* $z \in [\underline{v}, \overline{v}]$

$$\overline{v}\left(G^*(z) - G(z)\right) \leq p\int_{\underline{v}}^{z} \frac{t g^*(t)}{1 - pG^*(t)}(G(t) - G^*(t))dt, \tag{20}$$

then there is a symmetric equilibrium with the strategy given by

$$\beta(v) = \int_{\underline{v}}^{v} \frac{tg^*(t)}{1 - pG^*(t)} dt. \tag{21}$$

Condition (20) can be derived directly from (19) by starting with strategy (21) and imposing that, for any bid and value, G^* is the minimizing distribution. The condition implies a stochastic order which is weaker than FOSD when $p > 0$ and they are equivalent when $p = 0$. This stochastic order allows distributions in Δ to cross G^*. However, G^* must be sufficiently below the others on the lower segments of the support.

5. Example

In this section I provide an example where a symmetric, increasing equilibrium does not exist. Using Theorem 4.3, I construct a strategy which is the unique strategy that satisfies the necessary condition. For this strategy, the minimizing distribution for a given bid depends on the value of the player. The dependence is such that although the bid prescribed by the candidate strategy is a local maximum it is not a global maximum for some types.

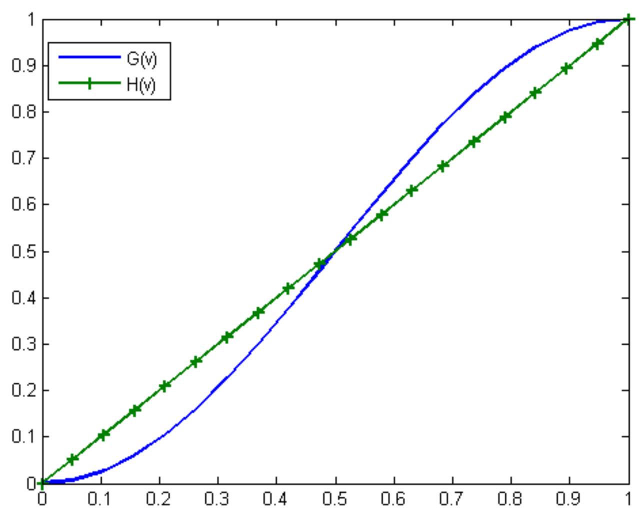

FIGURE 1. $\Delta = \{H(v) = v, G(v) = .5\sin(\pi(v - .5)) + .5\}$

Let the values come from the interval $[0, 1]$. Let $\Delta = \{G, H\}$ contain two distributions $G(v) = .5\sin(\pi(v - .5)) + .5$ and $H(v) = v$.[15] Furthermore, let $p = .9$.

[15] I could also let Δ be the convex hull of these two distributions; by the linearity of the expectations operator nothing would change.

To construct the unique candidate equilibrium strategy, suppose that β satisfies the conditions of Theorem 4.3. Define the strategy $\beta_F(v) \equiv \int_0^v \frac{tl(t)}{1-L(t)}dt$ for any distribution $L \in \Delta$. Since G is below H on the interval $(0, .5)$, $U(z; v, \beta, G) < U(z; v, \beta, H)$ for all $z \in (0, .5)$. This follows since the probability of winning is minimized and the expected payment is maximized by G when the player bids $z \in [0, .5]$. Since G minimizes the expected utility on $[0, .5]$, it follows from Theorem 4.3 that $\beta(v) = \beta_G(v)$ for all $v \in [0, .5]$. Define v^* as the lowest value such that $U(v^*; v^*, \beta_G, G) \leq U(v^*; v^*, \beta_G, H)$. This crossing is shown in Figure 2. Since G continues to minimize the expected utility on $[0, v^*]$, it follows from Theorem 4.3 that $\beta(v) = \beta_G(v)$ for all $v \in [0, v^*]$. In the proof of Proposition 5.1, I show that if β is an equilibrium, H is the minimizing distribution for a player with value above v^*. From Theorem 4.3, this implies that the only candidate for an equilibrium is the strategy defined in the following proposition.

Proposition 5.1. *The unique strategy that satisfies the condition of Theorem 4.3 is given by*

$$\beta(v) = \begin{cases} \beta_G(v) \text{ for } v \in [0, v^*] \\ \beta_G(v^*) - \beta_H(v^*) + \beta_H(v) \text{ for } v \in (v^*, 1]. \end{cases} \tag{22}$$

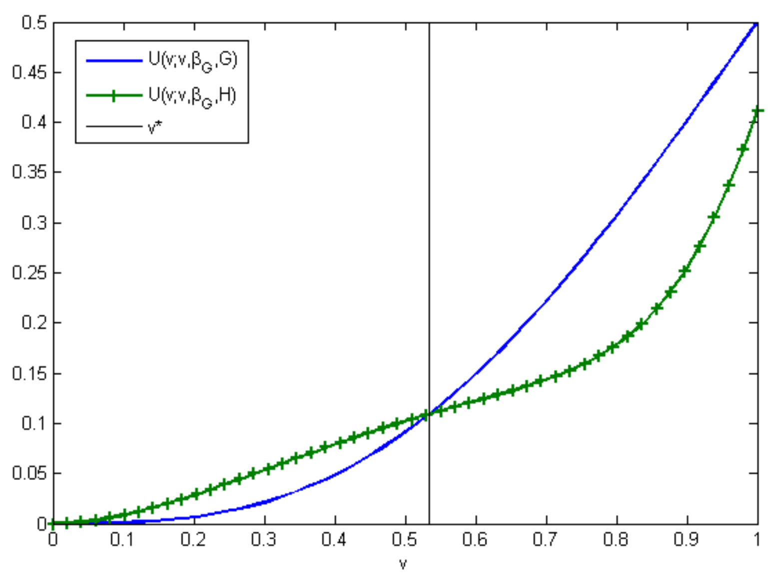

Figure 2. Definition of v^*

To see that there is no equilibrium, by straightforward calculation one can show that a player with value just above v^* strictly prefers to bid lower than $\beta(v^*)$, when the other player follows strategy β. The reason is that although a player with value v^* gets the same expected utility from and G and H at v^*, a player with a value just above v^* gets a strictly lower expected utility from H at v^*. This is the case because a player with a higher value cares more about the probability of winning. This means that for a player with value above v^* the MEU can be decreasing for transformed bids less than v^*. In this case, this provides a profitable deviation for some values. This situation is depicted in Figure 3. The local maximum of $U(z; v^* + .02, \beta, H)$ on the right is the MEU at the candidate equilibrium bid. However, the point of intersection of the two curves provides a higher utility. The source of the profitable deviation is that the minimizing distribution changes with the player's value such that the minimum expected utility is not quasi-concave.

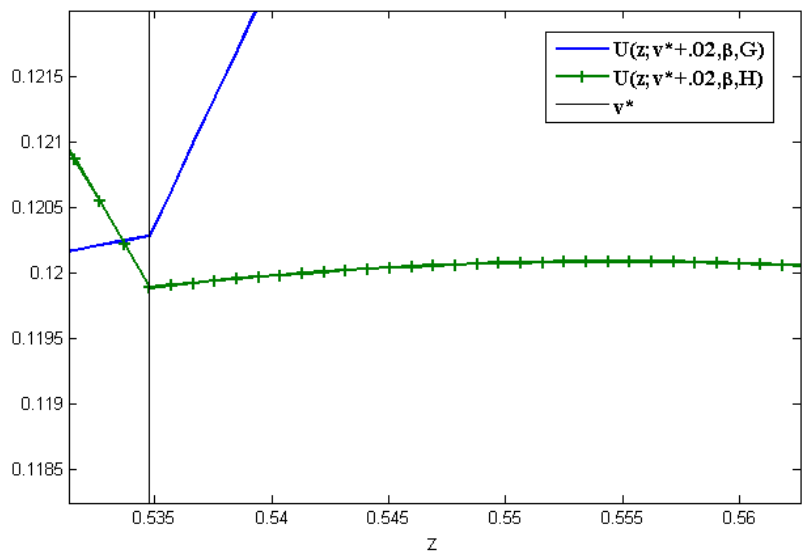

FIGURE 3. The minimum expected utility, $V(z; v^* + .02, \beta)$, is the minimum of the two curves depicted.

This example contrasts the results of a closely related paper by Bodoh-Creed[6]. He analyzes mechanism design problems with general ambiguous beliefs. If $\Delta \subset \tilde{\Delta}$ the reserve price for the revenue maximizing mechanism is lower for $\tilde{\Delta}$. Also, if an ex ante balanced budget bilateral trade mechanism maximizes the seller's revenue, efficient trade increases the more ambiguity the buyer and seller face. If a bilateral trade mechanism is efficient, increasing ambiguity will decrease the ex ante budget deficit. The observation is that increasing ambiguity improves efficiency in a mechanism design environment. In contrast, I show that a fixed mechanism may cease to have an efficient equilibrium if the players' ambiguity is increased.[16]

6. Extensions

6.1. **Smooth Ambiguity Aversion.** In the MEU model, I find that without making restrictive assumptions, there may not be an increasing, symmetric equilibrium. The reason for this result is that the minimizing distribution can depend on the value of the player in an endogenous way. This section argues that this result is robust to other specifications of ambiguity aversion. I extend the analysis to the smooth ambiguity aversion model formalized by Klibanoff, Marinacci, and Mukerji[20]. A comprehensive study of games with smooth ambiguity aversion is beyond the scope of this paper; however, I will show how the intuition gained from games with MEU applies to the smooth ambiguity aversion model.

To illustrate the potential issues involved, I describe the smooth ambiguity aversion model as it applies to the class of games studied above. Let $\Delta = \{F(.;\theta)\}_{\theta\in[0,1]}$ be a parametrized set of distributions on $[\underline{v}, \overline{v}]$ where F is measurable in θ. Define the expected utility

$$U_\theta(z;v) \equiv vF(z;\theta) - p\int_{\underline{v}}^{z} \beta(t)f(t;\theta)dt - (1-pF(z;\theta))\beta(z) \tag{23}$$

for an increasing strategy β. The player's interim utility from bidding like a player with value z is

$$W(z;v) = \int_0^1 \psi[U_\theta(z;v)]d\theta. \tag{24}$$

If $\psi : \mathbf{R} \to \mathbf{R}$ is concave, the player is said to be ambiguity averse. The concavity of ψ means that the lower expected utilities are weighted more heavily in the utility function. It is helpful to write the derivative of the utility function.

$$W'(z;v) = \int_0^1 \psi'[U_\theta(z;v)]\{vf(z;\theta) - (1-pF(z;\theta))\beta'(z)\}d\theta \tag{25}$$

$\psi'[U_\theta(z;v)]$ is a weighting function that weights distributions which yield low expected utility more heavily. Let $\phi(z,v) \equiv \psi'[U_\theta(z;v)]$ denote this weighting function and note that since the expected utility depends on the other player's strategy the weight depends on the strategy.

If a symmetric, increasing equilibrium, β, exists and solves the FOC,

$$\beta'(v) = \frac{\int_0^1 \phi(v;v)vf(v;\theta)d\theta}{\int_0^1 \phi(v;v)(1-pF(v;\theta))d\theta}. \tag{26}$$

[16] Also in the context of general equilibrium theory, ambiguity aversion tends to improve the efficiency of outcomes as shown by Castro, Pesce, and Yannelis[10].

Substituting (26) into (25) I get that

(27)

$$W'(z;v) = (\frac{\int_0^1 \phi(z;v)vf(z;\theta)d\theta}{\int_0^1 \phi(z;v)(1-pF(z;\theta))d\theta} - \frac{\int_0^1 \phi(z;z)zf(z;\theta)d\theta}{\int_0^1 \phi(z;z)(1-pF(z;\theta))d\theta}) \int_0^1 \phi(z;v)(1-pF(z;\theta))d\theta$$

To insure the quasi-concavity of W, one must make some assumptions regarding the parametrized family of distributions and ψ. In the special case that $p = 0$ and $\psi(x) = 1 - e^{-ax}$, (27) simplifies to the following expression which does not depend on the bid strategy.

$$W'(z;v) = \left(\frac{\int_0^1 e^{-avF(z;\theta)} vf(z;\theta)d\theta}{\int_0^1 e^{-avF(z;\theta)}d\theta} - \frac{\int_0^1 e^{-azF(z;\theta)} zf(z;\theta)d\theta}{\int_0^1 e^{-azF(z;\theta)}d\theta} \right) \int_0^1 \psi'[U_\theta(z;v)]d\theta. \tag{28}$$

This motivates the following proposition.

Proposition 6.1. *In the all-pay auction with $\psi(x) = 1 - e^{-ax}$, if*

$$\gamma(v,z) = \frac{\int_0^1 e^{-avF(z;\theta)} vf(z;\theta)d\theta}{\int_0^1 e^{-avF(z;\theta)}d\theta} \tag{29}$$

is increasing in v for all z, then

$$\beta(v) = \int_{\underline{v}}^{v} \frac{\int_0^1 e^{-atF(t;\theta)} tf(t;\theta)d\theta}{\int_0^1 e^{-atF(t;\theta)}d\theta} dt \tag{30}$$

is an equilibrium strategy.

The condition is analogous to the condition developed in Krishna and Morgan[23] in the context of affiliated values. The reason for assuming that $\gamma(v,z)$ is increasing, is to establish the quasi-concavity of equilibrium utility. Furthermore, in this particular case the sufficient condition depends only on the parameters, which include a and the parametrized set of distributions.

In general one might want

$$\hat{\gamma}(v,z) = \frac{\int_0^1 \phi(z;v)vf(z;\theta)d\theta}{\int_0^1 \phi(z;v)(1-pF(z;\theta))d\theta} \tag{31}$$

to be increasing in v for all z. However, it is not obvious what condition would insure this since the function ϕ depends on the strategy being played and thus is determined in equilibrium. Further work is needed to discover if there are more general conditions that can establish existence of an increasing equilibrium with the smooth ambiguity model.

The reasons for non existence of an increasing equilibrium in both the smooth ambiguity aversion model and MEU are similar. With smooth ambiguity aversion the weighting function ϕ depends on the player's value. MEU is simply the extreme case where weight is only given to the distributions which minimize the expected utility. In either case, restrictions on these weights are needed to insure the existence of an equilibrium. The main difficulty is that the weighting function is usually endogenous.

6.2. **Type Dependent Ambiguity.** The analysis can be extended to allow for ambiguity which depends on the value of the player. This can be used to model an environment where a player may believe that if his own value is relatively high the other player's value will tend to be high as well. To be formal suppose that $\Delta(v)$ is not constant in v. The necessary condition continues to hold as long as the following additional assumptions hold.

A. 5. $\Delta(v)$ is continuous in v.

A. 6. The conditions of A.2 hold for $\Delta(v)$ for all $v \in [\underline{v}, \overline{v}]$.

For the purposes of Theorem 4.3 and Proposition 4.4 it is sufficient that the set of minimizing distributions is upper semicontinuous and this additional assumption is sufficient for that conclusion. However, to write sufficient conditions for the existence of an increasing equilibrium one must take special care since the minimizing distribution will usually depend on the player's value.

To be more concrete, consider the following model of type dependent ambiguity aversion. Let Π be a set of symmetric joint distributions over the values of the two players. Π can be thought of as a set of priors that players have ex ante. Assume that the distributions in Π are twice differentiable in both arguments. For $F \in \Pi$, let $f(.|v)$ be the distribution of a player's value conditional on the other player's value being v. Let $\Delta(v) = \{G \in C_1([\underline{v}, \overline{v}])|g = f(.|v) \text{ for some } F \in \Pi\}$. This means that $\Delta(v)$ arises from prior by prior updating of the priors in Π.[17] From the discussion in Section 4, it seems natural to look for conditions under which there is a distribution $G_M \in \Pi$ such that $G_M(.|v)$ is the minimizing distribution in $\Delta(v)$ for any bid. In addition, as in Krishna and Morgan[23] one must be sure that $G_M(.|v)$ depends on v in such a way that an increasing equilibrium exists. The following proposition provides such conditions.

Proposition 6.2. *Suppose that there is a $G_M \in \Pi$ such that*

[17] There are many other ways to model how beliefs depend on v. The appropriate choice of updating rules is beyond the scope of this paper. For an axiomatization of the rule of updating each prior by Bayes rule see Pires[33]

(1) $G_M(.|v)$ *FOSD* $G(.|v)$ *for all* $G \in \Pi$ *and for all* $v \in [\underline{v}, \overline{v}]$
(2) *and*

$$\frac{vg_M(z|v)}{1 - pG_M(z|v)}$$

is increasing in v *for all* $z \in [\underline{v}, \overline{v}]$,

then there exists a symmetric, increasing equilibrium given by

$$\beta(v) = \int_{\underline{v}}^{v} \frac{tg_M(t|t)}{1 - pG_M(t|t)} \tag{32}$$

I give a brief outline of the proof as the argument is similar to the discussion in Section 4. Condition 1 of Proposition 6.2 guarantees that for any z or v, $G_M(.|v)$ will be an expected utility minimizing distribution. Using a straightforward generalization of Theorem 4.3, the only candidate equilibrium is given by (32). Condition 2 is sufficient to insure that the equilibrium MEU, which has the same form as equation (19), is maximized by the candidate equilibrium bid.

7. Conclusion

This paper discusses efficiency in a class of games with MEU. In sharp contrast to games with SEU, games with MEU may not have an increasing, symmetric equilibrium. This is because even though Δ, which is interpreted as the ambiguous belief, is independent of the value, the minimizing distribution may depend on the value. This can be resolved if Δ contains a worst distribution according to the relevant stochastic order. In that case there is an equilibrium in which the worst case distribution is the same for all values.

Some may see non existence of efficient equilibrium as a deficiency of the MEU model since increasing equilibria have many convenient properties: they are easy to understand for players, they can be easily characterized, and they are efficient. However, the non existence of an increasing equilibrium illustrates the complexity of games in which the expected utility of a bid does not depend solely on the probability of having the highest bid. When players with limited probabilistic knowledge need to weigh other aspects, such as the expected value of the other player's bid, it is not sufficient to only consider increasing strategies. The potential complexity and the inefficiency which results may explain why, when the mechanism can be chosen, mechanisms such as the first-price auction are prevalent.

It should be noted that this paper illustrates a method that can be used in other environments. For instance, the ambiguity could be made type dependent so that $\Delta(v)$ depends continuously on the type of the player. Here I study only monotone strategies. Thus, games with monotone equilibria in the SEU environment are candidates for this method. A straightforward generalization is the class of contests with spillovers described in Baye, Kovenock, and de Vries[5]. Also, games with a common value component as in Milgrom and Weber[28] can be studied in this way.

An interesting question for future research is the existence of an efficient equilibrium in other models of ambiguity aversion. Using the model of smooth ambiguity aversion by Klibanoff, Marinacci, and Mukerji[20], I provide an example in which assumptions based on the parameters of the model can be used to establish existence. More investigation is needed to see if the revenue rankings continue to apply. It would also be interesting, to study other models such as the Choquet expected utility[39] model used by Salo and Weber[38] to study the first-price auction.

Appendix A.

A.1. *Proof of Lemma 4.1:* Proceed by contradiction. Let β be an increasing equilibrium strategy and suppose that there exist $v, z \in [\underline{v}, \overline{v}]$ such that $|\beta(v) - \beta(z)| > M|v - z|$ where $M = \frac{\overline{vg}}{(1-p)^2}$. WLOG let $v > z$. The strategy subscript on the minimizing distributions is suppressed to avoid notational clutter.

$$V(v;v,\beta) - V(z;v,\beta) = U(v;v,\beta,G_{v,v}) - U(z;v,\beta,G_{v,z}) \leq U(v;v,\beta,G_{v,z}) - U(z;v,\beta,G_{v,z})$$

$$= v(G_{v,z}(v) - G_{v,z}(z)) - p\int_z^v \beta(t)g_{v,z}(t)dt - (1 - pG_{v,z}(v))\beta(v) + (1 - p(G_{v,z}(v) + G_{v,z}(z) - G_{v,z}(v)))\beta(z)$$

$$= v(G_{v,z}(v) - G_{v,z}(z)) - p\int_z^v \beta(t)g_{v,z}(t)dt - (1 - pG_{v,z}(v))(\beta(v) - \beta(z)) + p(G_{v,z}(v) - G_{v,z}(z))\beta(z)$$

$$\leq \overline{vg}(v - z) - (1 - p)(\beta(v) - \beta(z)) + p\overline{g}(v - z)\frac{\overline{v}}{1 - p}$$

$$= \frac{\overline{vg}}{1 - p}(v - z) - (1 - p)(\beta(v) - \beta(z)) < 0$$

The first inequality follows from the definition of $G_{v,z}$. The weak inequality on the fourth line follows because the densities are bounded by $\overline{g}$, and $\beta(v) \leq \frac{\overline{v}}{1-p}$ for all v. The bound on β follows since with probability $1-p$ each player pays his bid so it is a dominated strategy to bid above $\frac{\overline{v}}{1-p}$. The strict inequality follows from the first supposition. Thus any equilibrium is Lipschitz continuous with constant $M = \frac{\overline{vg}}{(1-p)^2}$. Because the last inequality is strict there is a positive measure of types that have a profitable deviation. This contradicts that β is an equilibrium.

□

Proof of Envelope Theorem: The proof uses the envelope theorems of Milgrom and Segal[27] (hereafter MS).

Part 1: $U(z; v, s, G)$ is absolutely continuous in z since each $G \in \Delta$ is absolutely continuous and s is absolutely continuous. By MS Theorem 2 to prove (1) it is sufficient to show that there exists $B > 0$ such that $|U'(z; v, s, G)| \leq B$ for almost all $z \in [\underline{v}, \overline{v}]$ and for all $G \in \Delta$.

Suppose $s(.)$ is differentiable at z. For all $G \in \Delta$,

$$|U'(z; v, s, G)| = |vg(z) - (1 - pG(z))s'(z)| \leq \frac{\overline{vg}}{(1-p)^2}. \tag{33}$$

The inequality follows since s is Lipschitz continuous with constant $M = \frac{\overline{vg}}{(1-p)^2}$ and g is bounded by $\overline{g}$. Since s is differentiable almost everywhere the result is proved.

The following definition is used in proving Part 2.

Definition A.1. The collection of functions $\{l(.; G)\}_{G \in \Delta}$ is equidifferentiable at z if

$$\frac{l(z'; G) - l(z; G)}{z' - z} \tag{34}$$

converges uniformly as $z' \to z$.

This condition is satisfied for instance if $\{l'(.; G)\}_{G \in \Delta}$ is an equicontinuous collection. Suppose $\{l(.; G)\}_{G \in \Delta}$ and $\{h(.; G)\}_{G \in \Delta}$ are equidifferentiable at v and $f(.)$ is differentiable at v. Then $\{l(.; G) + h(.; G)\}_{G \in \Delta}$ and $\{l(.; G)f(.)\}_{G \in \Delta}$ are equidifferentiable at v.

Part 2: From MS Theorem 3, it is sufficient to show that $\{U(.; v, s, G) : G \in \Delta\}$ is equidifferentiable at v wherever s is differentiable. Since Δ is compact in the sense that the densities form a compact set in $C([\underline{v}, \overline{v}])$, by the Arzelà-Ascoli theorem the densities of the distributions in Δ are an equicontinuous set. Thus, Δ is equidifferentiable. Since s is bounded and g is bounded above by $\overline{g}$ for all $G \in \Delta$, $\left\{\int_{\underline{v}}^{z} s(t)g(t)dt\right\}_{G \in \Delta}$ is equidifferentiable. $\{(1 - pG(.))s(.)\}_{G \in \Delta}$ is equidifferentiable at v whenever s is differentiable at v. Since equidifferentiability respects sums $\{U(.; v, s, G)\}_{G \in \Delta}$ is equidifferentiable at v whenever s is differentiable at v. Since s is differentiable almost everywhere the result follows from MS.

Part 3: is a direct consequence of Theorem 1 in MS.

□

Lemma A.2. *$\mathcal{G} : [\underline{v}, \overline{v}] \times [\underline{v}, \overline{v}] \times IBC([\underline{v}, \overline{v}]) \to 2^{\Delta}$ is upper semicontinuous, compact valued, and convex valued. Here $IBC([\underline{v}, \overline{v}])$ is the space of increasing, continuous functions on $[\underline{v}, \overline{v}]$ which are bounded by $\frac{\overline{v}}{1-p}$ and it is endowed with the uniform topology.*

Proof: First I show that

$$U(z;v,\beta,G) = vG(z) - \int_{\underline{v}}^{z} p\beta(t)g(t)dt - (1-pG(z))\beta(z) \tag{35}$$

is continuous in all of its arguments. Focus on the middle summand as continuity of the rest follows easily.

Let $|z-\hat{z}| < \epsilon_1$ and $\|g-\hat{g}\|_\infty < \epsilon_2$ and $\|\beta-\hat{\beta}\|_\infty < \epsilon_3$. WLOG let $z > \hat{z}$

$$\left|\int_{\underline{v}}^{z} p\beta(t)g(t)dt - \int_{\underline{v}}^{\hat{z}} p\hat{\beta}(t)\hat{g}(t)dt\right|$$

$$\leq \left|\int_{\underline{v}}^{\hat{z}} p\beta(t)g(t) - p\hat{\beta}(t)\hat{g}(t)dt\right| + \left|\int_{\hat{z}}^{z} p\beta(t)g(t)dt\right|$$

$$= \left|\int_{\underline{v}}^{\hat{z}} p\beta(t)g(t) - \hat{\beta}(t)g(t) + \hat{\beta}(t)g(t) - p\hat{\beta}(t)\hat{g}(t)dt\right| + \left|\int_{\hat{z}}^{z} p\beta(t)g(t)dt\right|$$

$$\leq \int_{\underline{v}}^{\hat{z}} pg(t)\left|\beta(t)-\hat{\beta}(t)\right|dt + \int_{\underline{v}}^{\hat{z}} p\hat{\beta}(t)\left|g(t)-\hat{g}(t)\right|dt + \left|\int_{\hat{z}}^{z} p\beta(t)g(t)dt\right|$$

$$< p\overline{g}\epsilon_3(\overline{v}-\underline{v}) + \frac{p\overline{v}}{(1-p)^2}\epsilon_2(\overline{v}-\underline{v}) + \frac{p\overline{v}\overline{g}}{(1-p)^2}\epsilon_1$$

By making ϵ_1, ϵ_2 and ϵ_3 small enough the result is obtained. Berge's maximum theorem establishes that $\mathcal{G}_{v,z;s} \equiv \arg\min_{G\in\Delta} U(v;z,s,G)$ is u.s.c. and compact valued.

That $\mathcal{G}_{v,b;s}$ is convex valued follows from the convexity of Δ and the linearity of $U(v;z,s,G)$ as a function of G.

□

Proof of Theorem 4.3: Let β, an increasing, symmetric equilibrium, be differentiable at v. It follows from the proof of the envelope theorem that $V'_+(.;v,\beta)$ and $V'_-(.;v,\beta)$ both exist at v. Since $\beta(v)$ is an equilibrium $V'_+(v;v,\beta) \geq 0 \geq V'_-(v;v,\beta)$. Furthermore, by MS Theorem 3

$$V'_+(v;v,\beta) = \lim_{z\to v+} vg_{v,z}(v) - (1-pG_{v,z}(v))\beta'(v) \tag{36}$$

and

$$V'_-(v;v,\beta) = \lim_{z\to v-} vg_{v,z}(v) - (1-pG_{v,z}(v))\beta'(v). \tag{37}$$

Since $\mathcal{G}_{v,z;\beta}$ is upper semicontinuous and since $vg(v) - (1-pG(v))\beta'(v)$ is a continuous function of G, there is a $\hat{G} \in \mathcal{G}_{v,z;\beta}$ such that $V'_+(v;v,\beta) = v\hat{g}(v) - (1-p\hat{G}(v))\beta'(v)$. There is also a $\overline{G} \in \mathcal{G}_{v,z;\beta}$ such that $V'_-(v;v,\beta) = v\overline{g}(v) - (1-p\overline{G}(v))\beta'(v)$. Since $\mathcal{G}_{v,z;\beta}$ is convex valued there exists a $\check{G} \in \mathcal{G}_{v,z;\beta}$ such that $v\check{g}(v) - (1-p\check{G}(v))\beta'(v) = 0$. Thus there is a selection $G_{v,z,\beta}$ of $\mathcal{G}_{v,z,\beta}$ such that $\beta'(v) = \frac{vg_{v,v;\beta}(v)}{1-pG_{v,v,\beta}(v)}$ a.e. $v \in [\underline{v}, \overline{v}]$. By Lemma4.1, β is absolutely continuous and can thus be written as in the theorem.

□

Background for proof of Proposition 4.4:

Definition A.3.

$$\lambda(v, z; s) \equiv \left\{ \lambda \in \mathbf{R} : \lambda = \frac{vg(z)}{1 - pG(z)} \text{ for some } G \in \mathcal{G}_{v,z,s} \right\} \tag{38}$$

Lemma A.4. *$\lambda(v, z; s)$ is upper semicontinuous and has compact, convex values.*

Proof:

$$\hat{\lambda}(v, G) \equiv \frac{vg(v)}{1 - pG(v)} \tag{39}$$

By the continuity of $\hat{\lambda}$ and Lemma A.2, $\lambda(v, b; s)$ is upper semicontinuous and compact valued. The continuity of $\hat{\lambda}(v, .)$ implies that $\lambda(v, b; s)$ is convex valued since $\mathcal{G}_{v,z,s}$ is convex valued. □

The proof of Proposition 4.4 uses a convergence result which is useful in the study of differential inclusions. For reference, I state a version of the theorem which is proved in Aubin and Cellina[4].

Proposition A.5. *[Convergence Theorem] Let F be a u.s.c. map from $\mathbf{R}^2$ to the closed, convex subsets of $\mathbf{R}$. Let I be an interval of $\mathbf{R}$ and $x_k(.)$ and $y_k(.)$ be measurable functions from I to $\mathbf{R}^2$ and $\mathbf{R}$, respectively, satisfying for almost all $t \in I$, for every ϵ-ball, $B_\epsilon(0)$, in $\mathbf{R}^2 \times \mathbf{R}$ there is a $k_0 \equiv k_0(t, \epsilon)$ such that for all $k \geq k_0$, $(x_k(t), y_k(t)) \in \text{graph}(F) + B_\epsilon(0)$.*

If

- *$x_k(.)$ converges almost everywhere to a function $x(.)$ from I to $\mathbf{R}^2$,*
- *$y_k(.)$ belongs to $L_1(I, \mathbf{R})$ and converges weakly to $y(.)$ in $L_1(I, \mathbf{R})$,*

then for almost all $t \in I$, $(x(t), y(t)) \in \text{graph}(F)$, i.e. $y(t) \in F(x(t))$.

Proof of Proposition 4.4:

The proof follows the technique in Aubin and Cellina (pages 128-129, [4]). For completeness and since the differential inclusion here is slightly different from theirs, I include the details. Let $M = \frac{\overline{v}\overline{g}}{(1-p)^2}$.

$$\mathcal{K} = \{x \in C([\underline{v}, \overline{v}]) : x \textit{ is Lipschitz with constant } M \textit{ and } x(\underline{v}) = 0\} \tag{40}$$

$\mathcal{K}$ is compact by the Arzelà-Ascoli Theorem.

$$\mathcal{J}(s) \equiv \{z \in \mathcal{K} : z'(v) \in \lambda(v, s(v); s)\} \tag{41}$$

A fixed point of $\mathcal{J}$ satisfies the conditions for the type of strategy described in the statement of the theorem.

I now show that the Kakutani-Glicksberg-Fan fixed point theorem applies to $\mathcal{J}(.)$. First I argue that $\mathcal{J}(.)$ is non empty. For any continuous $s(.)$, $\lambda(v, s(v); s)$ is u.s.c. as a function of v. Thus $\lambda(v, s(v); s)$ has a measurable selection. If w(.) is such a selection, then $\int_{\underline{v}}^{v} w(t)dt$ is in $\mathcal{J}(x)$. That $\mathcal{J}(.)$ is convex valued is straight forward since $\lambda(v, s(v); s)$ is convex valued.

To establish upper semicontinuity, since $\mathcal{K}$ is compact it is sufficient to show that $\mathcal{J}$ has a closed graph. This is done through the convergence theorem. Let $x_k \in \mathcal{K}$ and $z_k \in \mathcal{J}(x_k)$ be such that $x_k \to x$ and $z_k \to z$. Let $y_k = z_k'$ for all k. Since $||y_k||_\infty \leq M$ for all k, by the Banach-Alaoglu theorem there is a subsequence of $\{y_k\}$ and y such that $||y||_\infty \leq M$ and $\int_{\underline{v}}^{\overline{v}} y_k(t)\phi(t)dt \to \int_{\underline{v}}^{\overline{v}} y(t)\phi(t)dt$ for all $\phi \in L_1[\underline{v}, \overline{v}]$. Since $L_\infty[\underline{v}, \overline{v}]$ is a subset of $L_1[\underline{v}, \overline{v}]$, $\{y_k\}$ converges to y weakly as a sequence in $L_1[\underline{v}, \overline{v}]$.

Since y_k converges weakly it converges pointwise almost everywhere. $y_k(v) \in \lambda(v, x_k(v); x_k)$ so by the u.s.c. of λ in all of its arguments there exists a $k_0(v, \epsilon)$ s.t. for all $k \geq k_0$, $(x_k(v), y_k(v)) \in \text{graph}(\lambda(v, .; x)) + B_\epsilon(0)$ for almost all $v \in [\underline{v}, \overline{v}]$. By the convergence theorem $y(v) \in \lambda(v, x(v); x)$ for almost all $v \in [\underline{v}, \overline{v}]$.

Since $y = z'$, $J(.)$ is upper semicontinuous. By the Kakutani-Glicksberg-Fan fixed point theorem, there exists a strategy $\beta \in J(\beta)$. Such a strategy satisfies the conditions of the proposition.

□

Proof of Proposition 4.5: I will first prove that if β is the strategy played by the other player, then G^* is the worst case distribution. To that end, suppose that for some v and z in $[\underline{v}, \overline{v}]$ there is another distribution which gives a strictly lower expected utility. Using a similar expression to (19), the previous statement is equivalent to

$$\int_{\underline{v}}^{z} \left\{ \frac{vg^*(t)}{1 - pG^*(t)} - \frac{tg^*(t)}{1 - pG^*(t)} \right\} (1 - pG^*(t))dt > \int_{\underline{v}}^{z} \left\{ \frac{vg(t)}{1 - pG(t)} - \frac{tg^*(t)}{1 - pG^*(t)} \right\} (1 - pG(t))dt$$

For some $G \in \Delta$.

By canceling and collecting terms this implies

$$\overline{v}(G^*(z) - G(z)) > \int_{\underline{v}}^{z} tg^*(t) \left(1 - \frac{1 - pG(t)}{1 - pG^*(t)} \right) dt \tag{42}$$

However this contradicts the hypothesis about G^*. Since z and v where arbitrary G^* always minimizes the expected utility. So, β satisfies the necessary condition. For β, (19) reduces to

$$\int_{\underline{v}}^{z} (v-t)g^*(t)dt \tag{43}$$

which is clearly maximized at v. □

Proof of Proposition 5.1 I first show that, in a neighborhood above v^*, H is the minimizing distribution for any equilibrium strategy. By calculation, it is can be shown that H continues the be the minimizing distribution thereafter.

Observe that $h(v^*) < g(v^*)$ and $H(v^*) < G(v^*)$ together imply that $\beta'_H(v^*) < \beta'_G(v^*)$. Let β satisfy the conditions of Theorem 4.3 and let $\hat{\beta}$ be as in Proposition 5.1. By Theorem 4.3, $\beta_H(v) \leq \beta(v) \leq \beta_G(v)$ for all $v \in [v^*, v^{**}]$ where v^{**} is such that $h(v) < g(v)$ and $H(v) < G(v)$ both continue to hold on the interval. This implies that $U(v; v, \beta, G) \geq U(v; v, \beta_G, G)$ and $U(v; v, \hat{\beta}, H) \geq U(v; v, \beta, H)$ for all $v \in [v^*, v^{**})$. Define $\tilde{U}(v; s, F) \equiv U(v; v, s, F)$ for all $F \in \Delta$.

$$\tilde{U}'(v; \beta_G, G) = G(v) \geq H(v) = \tilde{U}'(v; \hat{\beta}, H) \tag{44}$$

for all $v \in (v^*, v^{**}]$. Since $U(v^*; v^*, \beta_G, G) = U(v^*; v^*, \hat{\beta}, H)$ and the expected utilities are absolutely continuous on $(v^*, v^{**}]$, $U(v^*; v^*, \beta_G, G) \geq U(v^*; v^*, \hat{\beta}, H)$ for all $v \in (v^*, v^{**}]$. Thus, H is the worst case distribution when both players follow β. By direct calculation v^{**} can be taken to be $\overline{v}$.

References

[1] Stephen Abbott. *Understanding analysis*. Springer New York, 2001.

[2] Erwin Amann and Wolfgang Leininger. Asymmetric all-pay auctions with incomplete information: The two-player case. *Games and Economic Behavior*, 14(1):1 – 18, 1996.

[3] Susan Athey. Single crossing properties and the existence of pure strategy equilibria in games of incomplete information. *Econometrica*, 69(4):861–869, 2001.

[4] J. Aubin and A. Cellina. *Differential Inclusions: Set-Valued Maps and Viability Theory*. Grundlehren Der Mathematischen Wissenschaften. Springer, 2012.

[5] Michael R. Baye, Dan Kovenock, and Casper G. de Vries. Comparative analysis of litigation systems: An auction-theoretic approach. *The Economic Journal*, 115(505):583–601, 2005.

[6] Aaron L. Bodoh-Creed. Ambiguous beliefs and mechanism design. *Games and Economic Behavior*, 2012.

[7] Subir Bose, Emre Ozdenoren, and Andreas Pape. Optimal auctions with ambiguity. *Theoretical Economics*, 1:411–438, 2006.

[8] Jeremy Bulow and Paul Klemperer. The generalized war of attrition. *The American Economic Review*, 89(1):pp. 175–189, 1999.

[9] Miguel Carvalho. Static vs dynamic auctions with ambiguity averse bidders. (2012-022), 2012.

[10] Luciano Castro, Marialaura Pesce, and Nicholas Yannelis. Core and equilibria under ambiguity. *Economic Theory*, 48(2/3):519 – 548, 2011.

[11] Daniel Ellsberg. Risk, ambiguity, and the savage axioms. *The Quarterly Journal of Economics*, pages 643–669, 1961.

[12] Gadi Fibich, Arieh Gavious, and Aner Sela. All-pay auctions with risk-averse players. *International Journal of Game Theory*, 34:583–599, 2006. 10.1007/s00182-006-0034-5.

[13] Drew Fudenberg and Jean Tirole. A theory of exit in duopoly. *Econometrica*, 54(4):pp. 943–960, 1986.

[14] Itzhak Gilboa and David Schmeidler. Maxmin expected utility with non-unique prior. *Journal of Mathematical Economics*, 18:141–153, 1989.

[15] Massimo Guidolin and Francesca Rinaldi. Ambiguity in asset pricing and portfolio choice: a review of the literature. *Theory and Decision*, 74(2):183–217, 2013.

[16] Lars Peter Hansen and Thomas J Sargent. *Robustness*. Princeton university press, 2008.

[17] Brent R. Hickman. Effort, race gaps, and affirmative action: A game-theoretic analysis of college admissions. *Mimeo*, 2010.

[18] John H. Kagel and Dan Levin. Individual bidder behavior in first-price private value auctions. *Economics Letters*, 19(2):125 – 128, 1985.

[19] John Kennan and Robert Wilson. Strategic bargaining models and interpretation of strike data. *Journal of Applied Econometrics*, 4(S1):S87–S130, 1989.

[20] Peter Klibanoff, Massimo Marinacci, and Sujoy Mukerji. A smooth model of decision making under ambiguity. *Econometrica*, 73(6):1849–1892, 2005.

[21] Kai A. Konrad and Dan Kovenock. Contests with stochastic abilities. *Economic Inquiry*, 48(1):89–103, 2010.

[22] Matthias Kräkel. Optimal risk taking in an uneven tournament game with risk averse players. *Journal of Mathematical Economics*, 44(11):1219 – 1231, 2008.

[23] Vijay Krishna and John Morgan. An analysis of the war of attrition and the all-pay auction,. *Journal of Economic Theory*, 72(2):343 – 362, 1997.

[24] Dan Levin and Emre Ozdenoren. Auctions with uncertain numbers of bidders. *Journal of Economic Theory*, 118(2):229–251, October 2004.

[25] Kin Chung Lo. Sealed bid auctions with uncertainty averse bidders. *Economic Theory*, 12(1):1–20, 1998.

[26] John Maynard Smith. The theory of games and the evolution of animal conflicts. *Journal of theoretical biology*, 47(1):209–221, 1974.

[27] Paul Milgrom and Ilya Segal. Envelope theorems for arbitrary choice sets. *Econometrica*, 70(2):583–601, 2002.

[28] Paul R. Milgrom and Robert J. Weber. A theory of auctions and competitive bidding. *Econometrica*, 50(5):pp. 1089–1122, 1982.

[29] Sujoy Mukerji. Ambiguity aversion and incompleteness of contractual form. *The American Economic Review*, 88(5):pp. 1207–1231, 1998.

[30] Roger B. Myerson. Optimal auction design. *Mathematics of Operations Research*, 6(1):58–73, 1981.

[31] Barry Nalebuff and John Riley. Asymmetric equilibria in the war of attrition. *Journal of Theoretical Biology*, 113(3):517 – 527, 1985.

[32] Barry J Nalebuff and Joseph E Stiglitz. Prizes and incentives: towards a general theory of compensation and competition. *The Bell Journal of Economics*, pages 21–43, 1983.

[33] Cesaltina Pacheco Pires. A rule for updating ambiguous beliefs. *Theory and Decision: An International Journal for Methods and Models in the Social and Decision Sciences*, 53(2):137–152, 09 2002. Last updated - 2012-08-10.

[34] Srgio O. Parreiras and Anna Rubinchik. Contests with three or more heterogeneous agents. *Games and Economic Behavior*, 68(2):703–715, March 2010.

[35] Brennan C. Platt, Joseph Price, and Henry Tappen. The role of risk preferences in pay-to-bid auctions. *Management Science*, 59(9):2117–2134, 2013.

[36] John G. Riley and William F. Samuelson. Optimal auctions. *The American Economic Review*, 71(3):381–392, 1981.

[37] Walter Rudin. *Principles of mathematical analysis*, volume 3. McGraw-Hill New York, 1964.

[38] Ahtia Salo and Martin Weber. Ambiguity aversion in first-price sealed-bid auctions. *Journal of Risk and Uncertainty*, 11(2):123–137, 1995.

[39] David Schmeidler. Subjective probability and expected utility without additivity. *Econometrica: Journal of the Econometric Society*, pages 571–587, 1989.

[40] Steven Stong. Ambiguity aversion in the all-pay auction. *Working Paper*, 2011.

[41] Toru Suzuki. Competitive problem solving and the optimal prize schemes. *Games and Economic Behavior*, 75(2):1009 – 1013, 2012.

[42] Gordon Tullock. Efficient rent seeking. In James Buchanan, Robert Tollison, and Gordon Tullock, editors, *Toward a Theory of the Rent Seeking Society*, pages 269–282. Texas A&M Press, 1980.

[43] Robert Weber. Auctions and competitive bidding. In *Proceedings of Symposia in Applied Mathematics*, volume 33, pages 143–170, 1985.